

Formation of photon spheres in boson stars with nonminimally coupled field

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Abstract

A static spherically symmetric asymptotically flat spacetime may allow for circular closed null-geodesics which are said to belong to a photon sphere. In the context of gravitational lensing in the strong deflection regime, the presence of a photon sphere leads to unbounded angle of deflection of light (multiple turns) and formation of relativistic images. In this paper we show that photon spheres may form in some configurations of boson stars constructed with free massive complex scalar field nonminimally coupled to gravity. Assuming the boson star is transparent to light, photon spheres would not only give raise to phenomena in the realm of strong gravitational lensing, but also to considerably increased photon flux in the central region of the star, relative to the flux in its surroundings.

1 Introduction

Deflection of light by a massive body, as predicted by General Relativity, led to one of the first experimental verifications of the theory [1]. Deflection of light is also the mechanism behind the rich phenomenology of gravitational lensing that has seen a wide range of applications in astrophysics [2]. In observationally relevant circumstances gravitational lensing occurs in the so called weak deflection regime; the light passes by a massive body at distances that are much greater than the Schwarzschild radius corresponding to the mass, and the overall angle of deflection is small. However, despite of known difficulties on the observational side, there is also considerable interest in gravitational lensing in the strong deflection regime, see e.g. [3] and references therein. This primarily includes the case of lensing by the Schwarzschild black hole [4, 5, 6], by the regular black hole [7], or other types of black holes, see [8] for a review. Light passing sufficiently close to the Schwarzschild black hole may, due to the presence of the photon sphere in the spacetime, suffer arbitrarily large deflection as it makes one or more complete turns around it before escaping to spatial infinity. In the context of gravitational lensing this leads to the

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formation of the so called relativistic images. Mathematically rigorous treatment of the concept of the photon sphere can be found in [9].

In this paper we look for a simple matter model that can be used to construct static, spherically symmetric self-gravitating bodies, that curve the spacetime in such a way that photon spheres are formed. In this regard one is faced with two possible routes. One is to construct a compact body, i.e. a body that is finite in the radial extent, with the surface lying below the photon sphere of the Schwarzschild spacetime. If such body is assumed to be non-transparent to photons, then its properties as a gravitational lens are equivalent to those of a Schwarzschild black hole. The other, more interesting route, is to allow the energy-momentum content to take up all space and assume that the matter it describes is transparent to light. The simplest matter model for which it is natural to assume that it does not interact electromagnetically is the scalar field. For example, lensing by configurations of massless scalar field with a naked spacetime singularity was studied in [10]. As we intend to show, photon spheres can form within the regular objects formed of scalar fields known as boson stars.

Boson stars were originally conceived as Klein–Gordon geons [11, 12], the self-gravitating configurations of the free massive scalar field minimally coupled to gravity, while among the extensions of the original model one finds quartic self-interaction [13], non-minimal coupling to gravity [14, 15], addition of the Brans–Dicke field [16], etc. Due to their simplicity boson stars proved to be an attractive theoretical model for studying properties of self gravitating objects in General Relativity. They were also considered as candidates for astrophysical objects such as supermassive Sgr A* at the centre of our galaxy [17]. Several reports on the subject appeared over the past decades [18, 19, 20]. Gravitational lensing by boson stars was considered in [21, 22]; relatively large angles of deflection of light forming primary and secondary images were found, but the possibility of formation of photon spheres and relativistic images was not reported so far.

The paper is organized as follows: In Section 2 we give a brief overview of the properties of null-geodesics in a static, spherically symmetric, asymptotically flat spacetime with photon spheres, and we introduce the notation that will be used through the rest of the paper. We also discuss a simple analytical model of a compact body with the surface within the Schwarzschild photon sphere. In Section 3 we construct boson stars from free massive complex scalar field nonminimally coupled to gravity and we show that photon spheres form some configurations. We sum up in Section 4.

2 Notation and a simple model

We will be considering the behaviour of null-geodesics in static, spherically symmetric, asymptotically flat spacetimes, free of singularities or black holes. The line element can be written using the coordinates $x^a = (t, r, \vartheta, \varphi)$ and geometrized units ($c = 1 = G$) as

$$ds^2 = g_{ab} dx^a dx^b = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2, \quad (1)$$

where the radial coordinate r is the area radius and $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ is the line element on the unit sphere. The Schwarzschild spacetime contains a black hole of mass $M > 0$ with the event horizon at $r = 2M$. Outside the black hole the metric components of the Schwarzschild spacetime are given by

$$e^{2\Phi(r)} = e^{-2\Lambda(r)} = 1 - 2M/r, \quad r > 2M, \quad (2)$$

M being the ADM mass of the spacetime. We will encounter the situation where the energy-momentum tensor vanishes at radii greater than some constant $R > 2M$ and where the metric components are given by (2) exactly, as well as situations where nonzero energy-momentum takes up all space and where (2) only represents the asymptotic form of the metric as $r \rightarrow \infty$.

The Lagrangian for the geodesics constrained to the $\vartheta = \pi/2$ plane can be written as

$$L = \sqrt{\pm (-e^{2\Phi(r)}\dot{t}^2 + e^{2\Lambda(r)}\dot{r}^2 + r^2\dot{\varphi}^2)}, \quad (3)$$

where the overdot denotes differentiation with respect to the parameter of the geodesic and the choice of the sign depends on whether the geodesic is spacelike (+) or timelike (-). As L does not depend on t and φ there are two constants of motion, $\partial L/\partial \dot{t}$ and $\partial L/\partial \dot{\varphi}$. The ratio of the two constants,

$$b = \frac{\partial L/\partial \dot{\varphi}}{\partial L/\partial \dot{t}} = -r^2 e^{-2\Phi(r)} \frac{d\varphi}{dt} = \text{const.}, \quad (4)$$

is proportional to the angular momentum of the test particle following a timelike geodesic and is known as the impact parameter. For a geodesic that can be extended to spatial infinity, i.e. to the region where the spacetime is essentially flat, the absolute value of the impact parameter can be interpreted as the distance between the considered geodesic and the purely radial geodesic that is (at spatial infinity) parallel to it.

By continuity, the impact parameter b defined in (4) is a constant of motion also for the null-geodesics, where in addition we have $L = 0$. Eliminating $d\varphi/dt$ one obtains an equation for dr/dt that can be written in the form of the nonrelativistic energy equation,

$$\frac{1}{2} \left(\frac{dr}{dt} \right)^2 = E - V(b, r), \quad (5)$$

where $E = 0$ and the ‘effective potential’ $V(b, r)$ is given by

$$V(b, r) = \frac{e^{2\Phi(r)-2\Lambda(r)}}{2} \left(\frac{b^2 e^{2\Phi(r)}}{r^2} - 1 \right). \quad (6)$$

In analogy with the familiar scenario from classical mechanics we see that null-geodesics exist only where $V(b, r) \leq 0$, as the r.h.s. of (5) must be non-negative for dr/dt to be real. If $V(b, r_0) = 0$ then at $r = r_0$ there is a turning point of $r(t)$ if $\partial V(b, r)/\partial r|_{r=r_0} \neq 0$, or an equilibrium point if $\partial V(b, r)/\partial r|_{r=r_0} = 0$. A turning point is an inner turning point (a minimum of $r(t)$) if $\partial V(b, r)/\partial r|_{r=r_0} < 0$; it is an outer turning point (a maximum of $r(t)$) if $\partial V(b, r)/\partial r|_{r=r_0} > 0$. An equilibrium point of $r(t)$ implies existence of closed (circular) null-geodesics along equators of the hypersurface $r = r_0$ which is known as the photon sphere. A photon sphere is said to be stable if $\partial^2 V(b, r)/\partial r^2 > 0$, or unstable if $\partial^2 V(b, r)/\partial r^2 < 0$. In the neighborhood of a stable photon sphere there are bound null geodesics with oscillatory $r(t)$ such that the outer turning point is above, and the inner turning point is below the photon sphere. Apart from circular null-geodesics, an unstable photon sphere implies existence of null-geodesics that asymptotically approach the photon sphere, either from the inner or from the outer side, making an infinite number of turns around it. It can be shown that the outermost photon sphere in an asymptotically flat static spherically symmetric spacetime is an unstable photon sphere.

To access the properties of null-geodesics in a given static spherically symmetric spacetime in a systematic way it is convenient to consider the relation among the impact parameter b and the radial coordinates of turning points or photon spheres r_0 . This relation follows from the condition $dr/dt|_{r=r_0} = 0$, or $V(b, r_0) = 0$, and reads

$$b(r_0) = \pm r_0 e^{-\Phi(r_0)}. \quad (7)$$

The double sign in (7) corresponds to the sign of the angular momentum, or the time direction one chooses along the given null-geodesic. For simplicity of notation, with no loss in generality, we will assume positive b in what follows. It can be easily shown that solutions to (7) with $db/dr_0 > 0$ represent inner turning points, while those with $db/dr_0 < 0$ represent outer turning points of null-geodesics; the solutions with $db/dr_0 = 0$ and $d^2b/dr_0^2 < 0$ represent stable, while those $d^2b/dr_0^2 > 0$ represent unstable photon spheres in the spacetime.

Let us first consider the region of the Schwarzschild spacetime outside the black hole where the metric components are given by (2). There is no solution to (7) with $r_0 > 0$ for impact parameter $b < 3^{3/2}M$. This implies that null-geodesics with $b < 3^{3/2}M$ extend to spatial infinity at one end, while their other end reaches the black hole horizon. At $b = 3^{3/2}M$ there is the unstable photon sphere with area radius $r_0 = 3M$ (see point A in Fig. 1). For $b > 3^{3/2}M$ one finds two turning points. The inner turning point with $r_0 > 3M$ and with the asymptote $r_0 = b - M$ as $b \rightarrow \infty$ represents the closest approach of the null-geodesics extending to spatial infinity. The outer turning point with $r_0 < 3M$ and with the asymptote $r_0 = 2M$ as $b \rightarrow \infty$ represents the outermost point reached by the geodesics with both ends at the black hole horizon.

We now proceed to replace the central segment of the Schwarzschild spacetime containing the black hole of mass M with an exact solution of the Einstein equations representing a static spherical body (compact object) of mass M and surface radius $R > 2M$. We choose to do so in such a way that the joining hypersurface $r = R$ lies inside the Schwarzschild photon sphere, i.e. we require $2M < R < 3M$. This implies that the ratio $2M/R$, known as the surface compactness of the body, belongs to the highly relativistic regime,

$$\frac{2}{3} < \frac{2M}{R} < 1. \quad (8)$$

Here one recalls Buchdahl's upper bound on the surface compactness for bodies involving isotropic pressures in their interior [23], $2M/R \leq 8/9$, and also that many stellar models develop disturbing features such as violation of the energy conditions or loss of dynamical stability at high values of surface compactness. However, simple models of massive bodies with sufficiently high surface compactness to satisfy the condition (8) can be found and we choose here to work with a model consisting of a massive spherical shell of infinitesimal thickness supported against gravity by its own surface pressure. It was shown in Ref. [24] that for the case of the flat metric in the shell interior this model allows the surface compactness as high as $24/25$ without the violation of the dominant energy condition. The metric components inside the shell are given by

$$e^{2\Phi(r)} = 1 - 2M/R, \quad e^{2\Lambda(r)} = 1, \quad r < R. \quad (9)$$

It is important to note that joining the metrics (2) and (9) at $r = R$ is not smooth. While $g_{tt} = -e^{2\Phi(r)}$ is continuous, $g_{rr} = e^{2\Lambda(r)}$ has a discontinuity at $r = R$. This feature is closely related to the δ -shaped distribution of matter of the massive shell along the radial

coordinate. The full technical details of the construction of the energy-momentum tensor on the hypersurface $r = R$ which involves the application of Israel's thin shell formalism [25] are not needed for the present discussion; we refer the interested reader to [24] or to [26, 27] where a similar model with a segment of the de Sitter spacetime in the interior, known as the gravastar, is worked out in detail.

The relation among the impact parameter b and the radial coordinate of the turning point or a photon sphere r_0 for the spacetime containing the massive shell obeying (8) is shown in Fig. 1. The curved line is the plot of (7) for the Schwarzschild metric (2) which is relevant for $r_0 > R$ (the part of this curve below the point B is shown dashed). The straight line with one end at $b = r_0 = 0$ and the other at B is the plot of (7) for the metric inside the shell (9) which is relevant for $r_0 < R$. The point A with the coordinates $r_0 = 3M$ and $b = 3^{3/2}M$ is the Schwarzschild photon sphere, while the part of the curve above A represents the inner turning points (radii of closest approach) for null-geodesics with $b > 3^{3/2}M$ reaching spatial infinity. These null-geodesics are entirely contained in the Schwarzschild segment of the spacetime and are in no way affected by the massive shell replacing the black hole. Part of the curve between A and B represents the outer turning points, and the part of the straight line between C and B represents the inner turning points, for bound null geodesics with impact parameter b in the range $3^{3/2}M < b < b_B = R/\sqrt{1 - 2M/R}$. Part of the straight line below C represents the inner turning points (radii of closest approach) for null-geodesics reaching spatial infinity with $b < 3^{3/2}M$. The turning points of these null-geodesics lie within the sphere of area radius

$$r_C = 3^{3/2}M\sqrt{1 - 2M/R}, \quad (10)$$

which is smaller than the radius of the shell. It is easy to see that $r_C \rightarrow 0$ for $2M/R \rightarrow 1$. This also means that there is no null-geodesic extending to spatial infinity with the radius of closest approach r_0 in the range $r_C \leq r_0 \leq 3M$. The behaviour of a bundle of null-geodesics that are parallel at spatial infinity is shown in Fig. 2 as it reaches the massive shell of radius $R = 5M/2$. The null-geodesics with the impact parameter $b < 3^{3/2}M$ (solid lines) show strong degree of focusing in the interior of the massive shell. This interesting feature could not be confirmed analytically since the null-geodesics in the region $r > R$ could only be constructed by numeric integration of (4) and (5).

In summary of the model we have considered, we refer to the points A , B and C in Fig. 1. Point A is the outer (unstable) photon sphere, while B is the inner (stable) photon sphere. Point C is defined so that $b_C = b_A$. The null geodesics extending to spatial infinity have turning points with $r_0 < r_C$ for impact parameter $b < b_A$, or with $r_0 > r_A$ for $b > b_A$. In addition, there is a class of bound null geodesics with impact parameter b in the range $b_A < b < b_B$ and the turning points r_0 in the range $r_C < r_0 < r_A$. All these qualitative features, i.e. the points A , B and C arranged as in Fig. 1, are to be found in a different context in the following Section.

3 Photon spheres in boson stars

The action involving the massive complex scalar field ϕ nonminimally coupled to gravity is

$$S = \int d^4x \sqrt{-g} \left(\left(\frac{1}{16\pi} + \xi \phi^* \phi \right) R - (\nabla^a \phi^*)(\nabla_a \phi) - \mu^2 \phi^* \phi \right), \quad (11)$$

where ξ is the nonminimal coupling constant, R is the curvature scalar and μ is the field mass. Since the action is invariant under the global phase transformation $\phi \rightarrow e^{i\epsilon} \phi$, there

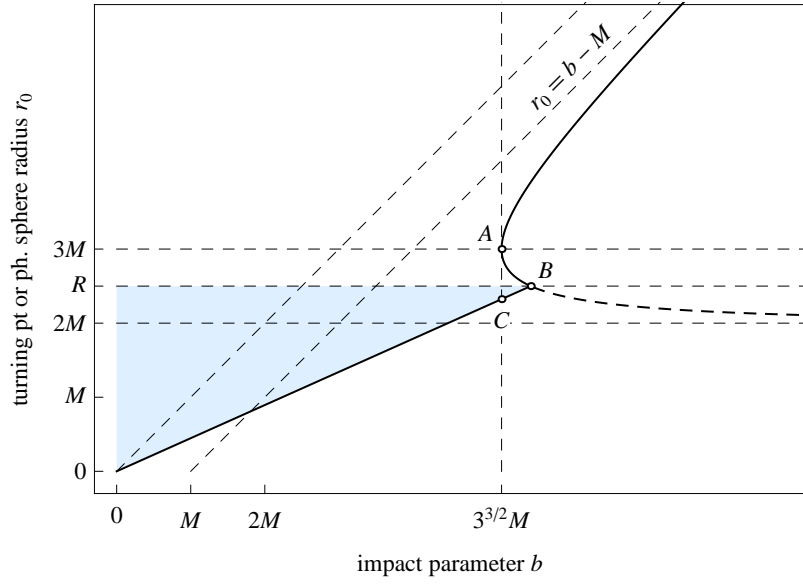


Figure 1: Turning point or photon sphere radius r_0 vs. impact parameter b (thick solid line) for null-geodesics in the spacetime containing the thin spherical shell of mass M and radius $R = \frac{5}{2}M$. (Null-geodesics exist only to the left of the thick solid line; shaded region corresponds to the shell interior.) Thick dashed line is the continuation of the r_0 vs. b curve for the case of black hole of mass M . Thin dashed lines are provided as eye-guides, the line $r_0 = b - M$ is the asymptote of $b(r_0)$ for $b \rightarrow \infty$.

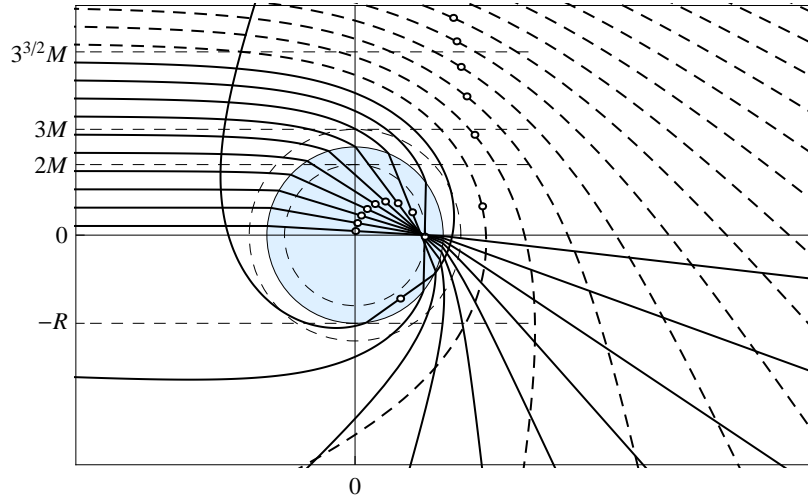


Figure 2: A bundle of asymptotically parallel null-geodesics passing by (impact parameter $b > 3^{3/2}M$, dashed lines) or through ($b < 3^{3/2}M$, solid lines) the spherical shell of mass M and radius $R = \frac{5}{2}M$ (thin solid circle with shaded interior). The turning point of each null-geodesic is indicated with the small circle.

is the conserved current, $j_a = i((\nabla_a \phi^*)\phi - (\nabla_a \phi)\phi^*)$, and the corresponding generator, the particle number N . The variation of (11) with respect to the metric gives the Einstein equation, $G_{ab} = 8\pi T_{ab}$, where $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ is the Einstein tensor and the energy-momentum tensor is given by

$$T_{ab} = \nabla_a \phi^* \nabla_b \phi + \nabla_a \phi \nabla_b \phi^* - g_{ab} (\nabla^c \phi^* \nabla_c \phi + \mu^2 \phi^* \phi) - 2\xi \phi^* \phi G_{ab} - 2\xi g_{ab} \nabla^c \nabla_c \phi^* \phi + 2\xi \nabla_a \nabla_b \phi^* \phi. \quad (12)$$

Note that the energy-momentum tensor can be written in a different way if the term $2\xi \phi^* \phi G_{ab}$ is transferred to the l.h.s. of the Einstein equation, which is then divided by $1 + 16\pi\xi \phi^* \phi$. There are also different approaches in the interpretation of the energy-momentum tensor in case of the nonminimally coupled scalar field, for a discussion see [28].

In spherical symmetry, with the metric written as in (1), the nontrivial components of the Einstein tensor are

$$G^t_t = r^{-2}(e^{-2\Lambda}(1 - 2r\Lambda') - 1), \quad (13)$$

$$G^r_r = r^{-2}(e^{-2\Lambda}(1 + 2r\Phi') - 1), \quad (14)$$

$$G^\vartheta_\vartheta = G^\varphi_\varphi = r^{-2}e^{-2\Lambda}((r\Phi' - r\Lambda')(1 + r\Phi') + r^2\Phi'') \quad (15)$$

(explicite notation of the r -dependence of functions is omitted and the prime denotes the r -derivatives). With the stationary Ansatz for the complex field, $\phi = \phi(r)e^{-i\omega t}$, where ϕ from here on denotes the real function, the nontrivial components of the energy-momentum tensor can be identified with the energy density ρ , radial pressure p and the transverse pressure q . They are given by

$$\rho = -T^t_t = (\mu^2 + e^{-2\Phi}\omega^2)\phi^2 + e^{-2\Lambda}\phi'^2 + 2\xi\phi^2 G^t_t + 4\xi e^{-2\Lambda}(\phi'^2 + (2/r - \Lambda')\phi'\phi + \phi''\phi), \quad (16)$$

$$p = T^r_r = (-\mu^2 + e^{-2\Phi}\omega^2)\phi^2 + e^{-2\Lambda}\phi'^2 - 2\xi\phi^2 G^r_r - 4\xi e^{-2\Lambda}(2/r + \Phi')\phi'\phi, \quad (17)$$

$$q = T^\vartheta_\vartheta = T^\varphi_\varphi = (-\mu^2 + e^{-2\Phi}\omega^2)\phi^2 - e^{-2\Lambda}\phi'^2 - 2\xi\phi^2 G^\vartheta_\vartheta - 4\xi e^{-2\Lambda}(\phi'^2 + (1/r - \Lambda' + \Phi')\phi'\phi + \phi''\phi). \quad (18)$$

Plugging (13)–(15) and (16)–(18) into the Einstein equation gives the system of three coupled ordinary differential equations with the three unknown functions: Φ , Λ and ϕ . The differential order of the system equals five since the highest derivatives that occur are Φ'' , Λ' and ϕ'' . The conservation condition $\nabla_a T^a_b = 0$, which gives

$$r^2 e^{2\Lambda}(-\mu^2 + e^{-2\Phi}\omega^2)\phi^2 + (2 - r\Lambda' + r\Phi')r\phi'\phi + r^2\phi''\phi = 2\xi(1 - e^{2\Lambda} + 2r\Phi' + r^2\Phi'^2 - r\Lambda'(2 + r\Phi') + r^2\Phi'')\phi^2, \quad (19)$$

can be used to eliminate Φ'' from the system. This reduces the differential order of the system by one. (Note that (19) is equivalent to what one obtains from (11) for the equation of motion for the scalar field.) The solution is determined with the boundary conditions which reflect the requirements that all functions are regular at $r = 0$, that the metric is asymptotically flat and coincides with (2), and that the field vanishes as $r \rightarrow \infty$. The number of the required boundary conditions equals the differential order

of the system (four) plus one since the system also involves the unknown frequency ω (eigenvalue) which has the role of an additional degree of freedom. The particle number is given with

$$N = \int d^3x \sqrt{-g} j^0 = \int_0^\infty 8\pi r^2 e^{\Lambda-\Phi} \omega \phi^2 dr \quad (20)$$

and can be obtained once the functions Φ , Λ and ϕ , and the frequency ω have been determined.

In order to set up the numerical procedure several further steps are taken. It is convenient to replace the metric function $\Lambda(r)$ with the so called ‘mass function’ $m(r)$ defined with

$$g_{rr} = e^{2\Lambda} = \frac{1}{1 - 2m/r} \quad (21)$$

(in the non-relativistic regime m is the mass within the sphere of radius r). The asymptotic value of m as $r \rightarrow \infty$ is the ADM mass of the spacetime, M . Radial coordinate $\tilde{x} = \mu r$ is introduced which is then mapped onto the compact domain with the transformation $x = \tilde{x}/(1 + \tilde{x}) \in [0, 1]$. Following the usual practice in the literature we introduce the field variable $\sigma = \sqrt{8\pi}\phi$. Finally, for the boundary conditions at $x = 0$ (corresponding to $r = 0$) we choose $m = 0$, $\sigma = \sigma_0$ and $\sigma' = 0$, while at $x = 1$ (corresponding to $r \rightarrow \infty$) we choose $\Phi = 0$ and $\sigma = 0$. The central value of the field variable, $\sigma_0 = \sigma(0)$, is used to parametrize the family of solutions obtained for some ξ ; for each ξ and σ_0 , the solution consists of the three functions, Φ , m and σ , and the eigenvalue ω . We used the COLSYS boundary value code [29] and checked that our procedure fully reproduces the results presented in the Table 1 of Ref. [14].

We limited our search for the photon spheres in the spacetimes sourced by the boson stars to the configurations without nodes in ϕ . (The configurations with one or more nodes are sometimes referred to as the excited boson stars [30].) For a given value of ξ , as the central value of the field variable σ_0 increases starting from $\sigma_0 = 0$, the mass M and the particle number N also increase from zero up to the (first) critical configuration where they reach their maxima. In analogy with the similar behaviour of mass in fluid stars, the first maximum of M suggests the onset of dynamical instability. In the case of minimal coupling of the scalar field and gravity ($\xi = 0$) the loss of stability with respect to radial perturbations at the first critical configuration has been confirmed by perturbative analysis [31, 32] and also by other methods [33, 34]. As σ_0 increases beyond the first critical configuration, the mass M and the particle number N go through a sequence of minima and maxima resembling strongly damped oscillations about respective asymptotic values. Subsequent maxima of M and N , that will also be referred to as critical configurations, are lower than their maxima at the first critical configuration.

The binding energy of a boson star can be defined as

$$E_b = M - \mu N. \quad (22)$$

For all values of ξ that we considered we found that the critical configurations (maxima of M and N) correspond to the minima of E_b . In all first critical configurations the binding energy E_b is negative which means that the bosons in such configurations may not disperse to infinity without the input of additional energy. For negative ξ and up to $\xi = 2$, as σ_0 increases beyond the first critical configuration, E_b becomes positive before the second critical configuration is reached and remains positive with the further increase of σ_0 . Interestingly, for $\xi = 4$ and greater, at the second critical configuration E_b is negative. The dependence of the binding energy (22) on σ_0 for some values of ξ

is shown in Fig. 3; one can see that for $\xi = 4$ there is a positive binding energy barrier between the first and the second critical configuration, while for sufficiently large values of ξ this barrier disappears. It should be emphasized, however, that negative E_b at some equilibrium configuration is a common characteristic of a gravitationally bound states, but does not by itself imply stability.

As outlined in the preceding section, in a given spacetime the photon spheres can be located by examining the behaviour of the function $b(r_0)$ defined in (7). If $b(r_0)$ is monotonically increasing with $r_0 \geq 0$, there is no photon sphere in the spacetime, whereas if extrema exist, they correspond to photon spheres. In the present context a more robust procedure for locating the photon spheres can be formulated. The condition for the extremum of $b(r_0)$ can be written as

$$0 = \frac{db}{dr_0} = e^{-\Phi}(1 - r\Phi'), \quad (23)$$

where Φ' can be expressed in terms of the mass function m and the radial pressure p using the well known equation

$$\Phi' = \frac{m + 4\pi r^3 p}{r^2(1 - 2m/r)} \quad (24)$$

(in the non-relativistic regime Φ is the gravitational potential). It follows that the condition for the existence of the photon sphere in the spacetime can be expressed as

$$\frac{2m}{r} = \frac{2}{3}(1 - 4\pi r^2 p). \quad (25)$$

The quantity on the l.h.s. is known as the ‘compactness function’ and is always less than unity, while the r.h.s. is less than two-thirds if the radial pressure is positive. It is interesting to observe that when one considers the compact object of mass M and surface radius R , the condition (8) for the photon sphere in the vacuum segment of the spacetime requires the surface compactness $2M/R$ to be greater than $2/3$; here we find that, for positive radial pressure p , the compactness function may satisfy the condition (25) at values that are less than $2/3$.

In the range of values of the non-minimal coupling constant ξ that we considered (see Table 1), we did not find photon spheres at or below the value of σ_0 corresponding to the first critical configuration. Interestingly, the photon spheres were found in the second critical configurations for $\xi = 0$ and above. However, only at $\xi = 4$ and above we have negative binding energy at the second critical configuration. Fig. 4 shows the relation among the turning point radius r_0 and the impact parameter b for the boson star with $\xi = 4$. Following the notation scheme introduced in the preceding section, the outer (unstable) photon sphere is marked with A and the inner (stable) photon sphere is marked with B . The impact parameter corresponding to the photon sphere A is denoted with b_{ps} . The area radius of the sphere containing the turning points of all null-geodesics with $b < b_{ps}$ is denoted with r_C . The layout of the points A , B and C for higher values of ξ is very similar to the layout shown in Fig. 4. Fig. 5 shows how the condition (25) can be used to locate the photon spheres.

In the simple model of the spacetime with photon spheres constructed in the preceding section we found strong degree of focusing of null-geodesics in the interior of the compact object. Here, as a measure of shrinking of the bundle of null-geodesics that are initially parallel at spatial infinity, one can consider the ratio of the two proper areas defined as

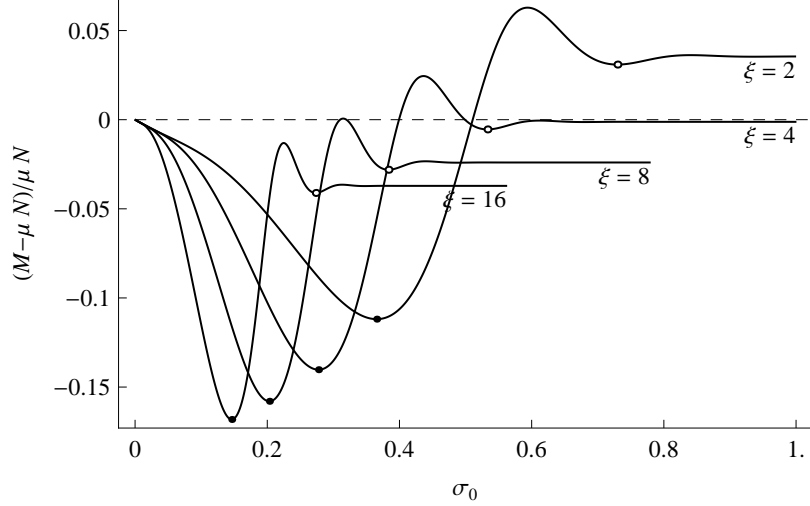


Figure 3: Binding energy (22) of the boson star per particle in units of the boson mass μ vs. the central value of the field variable σ_0 for the nonminimal coupling constant $\xi = 2, 4, 8, 16$. First and second critical configurations are indicated with filled and hollow circles, respectively.

follows. The first is the proper area of the circular cross section of the part of the bundle at spatial infinity containing all null geodesics that (in the course of time) pass through the photon sphere and enter its interior. These geodesics must have the impact parameter $b < b_{\text{ps}}$ so the proper area of the cross section is $A_\infty = b_{\text{ps}}^2 \pi$. The second proper area is that of the sphere containing the turning points of the geodesics that enter the photon sphere. As the area-radius of this sphere is r_C , its proper area is $A_C = 4r_C^2 \pi$. Defined in this way, the null geodesics that pass through A_∞ also pass through A_C , and vice versa. The ratio of the two proper areas is

$$\frac{A_\infty}{A_C} = \frac{b_{\text{ps}}^2}{4r_C^2}. \quad (26)$$

It can be interpreted as the ratio of the number of null-geodesics passing through unit proper area at $r = r_C$ and at $r \rightarrow \infty$. As it can be seen from the figures in Table 1, in the configurations of boson stars where we found photon spheres this ratio is by several orders of magnitude greater than unity. If one also considered the number of null-particles (photons) passing through unit proper area in unit proper time, assuming steady flux at $r \rightarrow \infty$, an additional factor of $e^{-\Phi(r_C)} = b_{\text{ps}}/r_C$ would enter due to the difference in clock ticking rates.

4 Conclusions

We have shown that photon spheres are present in some configurations of the boson stars constructed with the free massive scalar field nonminimally coupled to gravity. This implies the existence of null-geodesics that make arbitrarily many turns around the central region of the star before escaping to spatial infinity. Assuming the boson star is transparent to light, this further leads to the possibility of formation of the relativistic images in the strong deflection regime of gravitational lensing. With the finding of the

Table 1: Central value of the field variable σ_0 , mass M and the particle number N of the first and the second critical configuration of the boson star obtained with a sequence of values of the nonminimal coupling constant ξ . The impact parameter corresponding to the photon sphere b_{ps} , area radius of the photon sphere r_A , and the area radius r_C of the sphere containing the turning points for null-geodesics with $b < b_{\text{ps}}$ are given for configurations where the photon sphere exists. M_{P} denotes the Planck mass.

ξ	$(\sigma_0)_{\text{crit.}}$	$M/(M_{\text{P}}^2/\mu)$	$N/(M_{\text{P}}^2/\mu^2)$	b_{ps}/M	r_A/M	r_C/M
16	0.146287	2.89786	3.48375			
	0.273464	2.18529	2.27896	3.50852	0.196796	0.0233624
8	0.203279	2.05492	2.44047			
	0.383436	1.54653	1.59122	3.47565	0.197621	0.0236546
4	0.277568	1.46454	1.70347			
	0.533207	1.09811	1.10413	3.41338	0.199297	0.0242506
2	0.365541	1.05839	1.19175			
	0.729797	0.78796	0.764328	3.29866	0.202721	0.0254954
0	0.271059	0.633001	0.653003			
	1.62527	0.382621	0.330516	1.74432	0.176402	0.134695

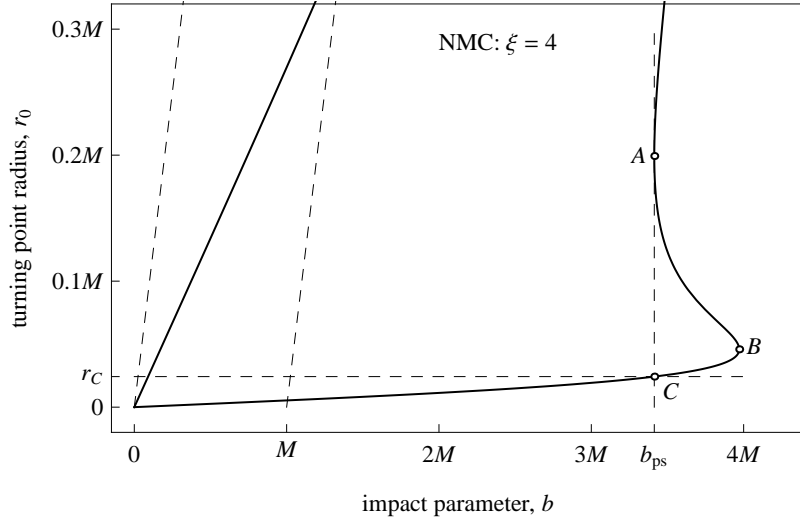


Figure 4: Radius of the turning point (or photon sphere) r_0 vs. impact parameter b for the null-geodesics in the spacetime containing the boson star with the nonminimal coupling constant $\xi = 4$ at the first (no photon spheres) and at the second (photon spheres A and B) critical configurations. The impact parameter corresponding to the photon sphere A is indicated with b_{ps} , the area-radius of the sphere containing the turning points for null-geodesics with $b < b_{\text{ps}}$ is indicated with r_C .

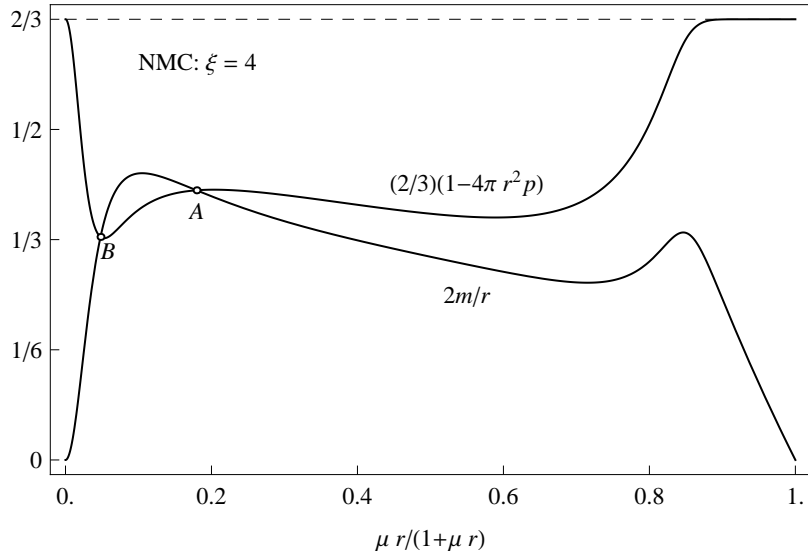


Figure 5: Locating of the photon spheres A and B in the spacetime containing the boson star with the nonminimal coupling constant $\xi = 4$ at the second critical configuration as the solutions of (25).

photon spheres within the boson stars, the scalar field appears to be a suitable matter model for modeling of strong gravitational lenses that do not involve singularities of black holes in the spacetime. Another interesting feature of global (no black holes) spherically symmetric spacetimes with photon spheres, relative to those with black holes, is the high degree of focusing of the incident light into the small central region of the spacetime. For the boson stars we have considered we were able to give a measure of contraction of the incident bundle of parallel null-geodesics in form of the ratio (26).

However, one must not overlook the not so desirable properties of the configurations of boson stars in which photon spheres were found. In the case of the minimally coupled field ($\xi = 0$) photon spheres were found in configurations that are known to be dynamically unstable and that have positive binding energy (positive work is required to ‘compress’ the bosons with initially zero kinetic energy at spatial infinity into the equilibrium configuration). As we have shown, the unusual sign of the binding energy of the configurations with photon spheres can be reversed with the introduction of sufficiently strong nonminimal coupling of the scalar field to gravity, but the issue of instability remains open. In this regard it would be interesting to carry out the linearized perturbation analysis for the boson stars with nonminimally coupled field (e.g. like it has been done for $\xi = 0$ in [32]) and if unstable modes are found in the configurations with photon spheres, to obtain the timescales of the growing perturbations. If these are found to be long relative to some time scale reflecting the size of the spacetime region in which gravitational lensing is taking place, then the boson stars with photon spheres would not be ruled out on the basis of instability as models of massive dark objects giving raise to phenomena of strong gravitational lensing. The possible extension of this work could be to explicitly study the angular pattern, magnifications and the time delays of relativistic images formed by boson stars and compare them with the results that are obtained when the massive dark objects in the centres of nearby galaxies are modeled as black holes [6, 35].

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References

- [1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. Wiley, 1972.
- [2] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses*. Springer-Verlag, 1992.
- [3] E. F. Eiroa, “Strong deflection gravitational lensing,” *ArXiv e-prints* (2012) , [arXiv:1212.4535 \[gr-qc\]](#).
- [4] H. C. Ohanian, “The black hole as a gravitational “lens”,” *Am. J. Phys.* **55** (1987) 428.
- [5] K. S. Virbhadra and G. F. R. Ellis, “Schwarzschild black hole lensing,” *Phys. Rev. D* **62** (2000) 084003, [arXiv:astro-ph/9904193](#).
- [6] K. S. Virbhadra, “Relativistic images of Schwarzschild black hole lensing,” *Phys. Rev. D* **79** (2009) 083004, [arXiv:0810.2109 \[gr-qc\]](#).
- [7] E. F. Eiroa and C. M. Sendra, “Gravitational lensing by a regular black hole,” *Class. Quantum Grav.* **28** (2011) 085008, [arXiv:1011.2455 \[gr-qc\]](#).
- [8] V. Bozza, “Gravitational lensing by black holes,” *Gen. Rel. Grav.* **42** (2010) 2269, [arXiv:0911.2187 \[gr-qc\]](#).
- [9] C.-M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, “The geometry of photon surfaces,” *J. Math. Phys.* **42** (2001) 818, [arXiv:gr-qc/0005050](#).
- [10] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, “Role of the scalar field in gravitational lensing,” *Astron. Astrophys.* **337** (1998) 1, [arXiv:astro-ph/9801174](#).
- [11] D. J. Kaup, “Klein-Gordon Geon,” *Phys. Rev.* **172** (1968) 1331.
- [12] R. Ruffini and S. Bonazzola, “Systems of Self-Gravitating Particles in General Relativity and the Concept of an Equation of State,” *Phys. Rev.* **187** (1969) 1767.
- [13] M. Colpi, S. L. Shapiro, and I. Wasserman, “Boson stars - Gravitational equilibria of self-interacting scalar fields,” *Phys. Rev. Lett.* **57** (1986) 2485.
- [14] J. J. van der Bij and M. Gleiser, “Stars of bosons with non-minimal energy-momentum tensor,” *Phys. Lett. B* **194** (1987) 482.
- [15] D. Horvat and A. Marunović, “Dark energy-like stars from nonminimally coupled scalar field,” *ArXiv e-prints* (2012) , [arXiv:1212.3781 \[gr-qc\]](#).
- [16] D. F. Torres, F. E. Schunck, and A. R. Liddle, “Brans - Dicke boson stars: configurations and stability through cosmic history,” *Class. Quantum Grav.* **15** (1998) 3701, [arXiv:gr-qc/9803094](#).

- [17] D. F. Torres, S. Capozziello, and G. Lambiase, “Supermassive boson star at the galactic center?,” *Phys. Rev. D* **62** (2000) 104012, [arXiv:astro-ph/0004064](#).
- [18] P. Jetzer, “Boson stars,” *Phys. Rep.* **220** (1992) 163.
- [19] F. E. Schunck and E. W. Mielke, “General relativistic boson stars,” *Class. Quantum Grav.* **20** (2003) R301.
- [20] S. L. Liebling and C. Palenzuela, “Dynamical Boson Stars,” *Living Rev. Relativity* **15** (2012) 6, [arXiv:1202.5809 \[gr-qc\]](#).
- [21] M. P. Dabrowski and F. E. Schunck, “Boson Stars as Gravitational Lenses,” *Astrophys. J.* **535** (2000) 316, [arXiv:astro-ph/9807039](#).
- [22] A. Y. Bin-Nun, “Method for detecting a boson star at Sgr A* through gravitational lensing,” *ArXiv e-prints* (2013) , [arXiv:1301.1396 \[gr-qc\]](#).
- [23] H. A. Buchdahl, “General Relativistic Fluid Spheres,” *Phys. Rev.* **116** (1959) 1027.
- [24] J. Frauendiener, C. Hoenselaers, and W. Konrad, “A shell around a black hole,” *Class. Quantum Grav.* **7** (1990) 585.
- [25] W. Israel, “Singular Hypersurfaces and Thin Shells in General Relativity,” *Nuovo Cimento B* **44** (1966) 1.
- [26] M. Visser and D. L. Wiltshire, “Stable gravastars - an alternative to black holes?,” *Class. Quantum Grav.* **21** (2004) 1135, [arXiv:gr-qc/0310107](#).
- [27] D. Horvat and S. Ilić, “Gravastar energy conditions revisited,” *Class. Quant. Grav.* **24** (2007) 5637, [arXiv:0707.1636 \[gr-qc\]](#).
- [28] S. Bellucci and V. Faraoni, “Energy conditions and classical scalar fields,” *Nucl. Phys. B* **640** (2002) 453, [hep-th/0106168](#).
- [29] U. Asher, J. Christiansen, and R. D. Russel, “Collocation Software for Boundary-Value ODEs,” *ACM Trans. Math. Softw.* **7** (1981) 209.
- [30] P. Jetzer, “Stability of excited bosonic stellar configurations,” *Phys. Lett. B* **222** (1989) 447.
- [31] T. D. Lee and Y. Pang, “Stability of mini-boson stars,” *Nucl. Phys. B* **315** (1989) 477.
- [32] M. Gleiser and R. Watkins, “Gravitational stability of scalar matter,” *Nuclear Physics B* **319** (1989) 733.
- [33] F. V. Kusmartsev, E. W. Mielke, and F. E. Schunck, “Gravitational stability of boson stars,” *Phys. Rev. D* **43** (1991) 3895, [arXiv:0810.0696 \[astro-ph\]](#).
- [34] S. H. Hawley and M. W. Choptuik, “Boson stars driven to the brink of black hole formation,” *Phys. Rev. D* **62** (2000) 104024, [arXiv:gr-qc/0007039](#).
- [35] V. Bozza and L. Mancini, “Time Delay in Black Hole Gravitational Lensing as a Distance Estimator,” *Gen. Relat. Grav.* **36** (2004) 435, [arXiv:gr-qc/0305007](#).